



Non-Fermi-liquid behavior in neutral bilayer graphene

Yafis Barlas and Kun Yang

National High Magnetic Field Laboratory and Department of Physics, The Florida State University, Tallahassee, Florida 32306, USA
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We calculate the density-density response function and electron self-energy for undoped bilayer graphene within the random phase approximation. We show that the quasiparticle decay rate scales linearly with the quasiparticle energy, and quasiparticle weight vanishes logarithmically in the low-energy limit, indicating non-Fermi-liquid behavior. This is a consequence of the absence of a Fermi surface for neutral bilayer graphene and corresponding larger phase space available for scattering processes. Experimental consequences of our results as well as their differences from those of single-layer graphene are discussed.

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I. INTRODUCTION

The isolation and subsequent identification of graphene, an atomically thin electron system, has led to intense experimental and theoretical interest.¹ Recent experimental progress¹ has also led to techniques that enable isolation and study of systems with a small number of graphene layers, of particular importance is *AB*-Bernal stacked bilayer graphene,² a system which shares some features both with graphene and two-dimensional electron gas (2DEG),³ however at the same time different from both. Neglecting trigonal warping balanced bilayer graphene can be identified as a zero-gap semiconductor with quadratic dispersion; for undoped bilayer graphene the Fermi energy lies at the neutral Fermi point, described as the point where the degenerate particle-hole bands meet. Collectively these systems can be classified as chiral 2DEGs.⁴ Electron-electron interactions in chiral 2DEGs can lead to interesting quasiparticle properties, for example quasiparticle velocity enhancement⁵ in graphene due to the presence of unscreened Coulomb interactions. Most of the physics in this Rapid Communication focuses on the difference in graphene and bilayer graphene's chiral 2DEG.

In this Rapid Communication we investigate the quasiparticle properties of undoped bilayer graphene due to electron-electron interactions. Short-ranged interactions for quadratic dispersion in two dimensions are marginal⁶⁻⁸ at the tree level while Coulomb interactions are relevant; this already points to the possibility of non-Fermi-liquid behavior. Unlike neutral graphene where electron-electron interactions are unscreened, Coulomb interactions in bilayer graphene are screened due to the presence of a finite density of states, and dynamically generate a momentum scale $\lambda = me^2/\epsilon$ proportional to the inverse Thomas-Fermi screening length $q_{TF} = 4\lambda \log[4]$. Below this scale this system resembles one with effective short-ranged interactions. Based on the scaling form of the density-density response function we demonstrate within the random phase approximation (RPA) that there is *no* renormalization to the electron effective mass, while the imaginary part of the electron self-energy $\text{Im}\Sigma \sim \omega$ below the screening scale. As a result the quasiparticle has a logarithmically vanishing spectral weight at low energies, indicative of non-Fermi-liquid behavior. This is very different from a 2DEG where the phase space available for scattering is limited by the energy thereby giving energy

squared dependence for the quasiparticle decay rate, and the quasiparticle weight remains finite at the Fermi surface. We argue this non-Fermi-liquid behavior is a consequence of the absence of a Fermi surface for neutral bilayer graphene and corresponding larger phase space available for scattering processes, which should be a robust result beyond RPA. The long-wavelength behavior of the electron spectral function $A(k, \omega) = A(k, -\omega)$ is plotted in Fig. 1, which can be compared with angle-resolved photoemission spectroscopy measurements currently being employed to study the effects of interactions in graphene systems.⁹

II. BILAYER GRAPHENE EFFECTIVE MODEL

The low-energy properties of Bernal stacked bilayer graphene can be adequately described by quasiparticles with parabolic dispersion³ exhibiting a Berry phase of 2π . Ne-

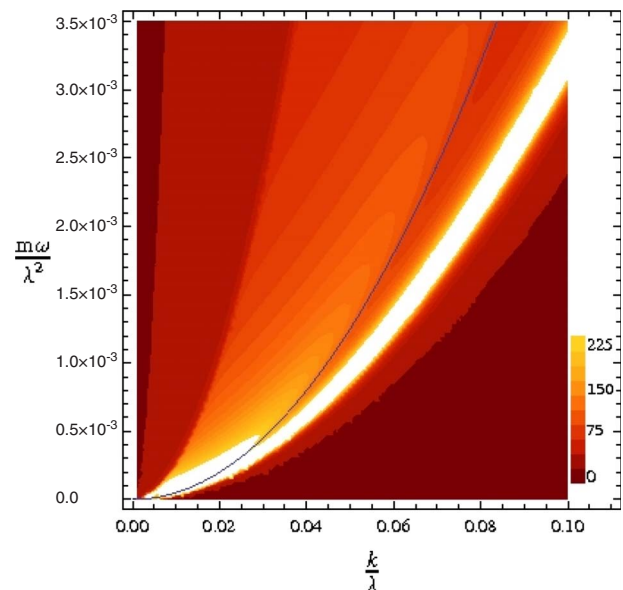


FIG. 1. (Color online) Intensity plot of electron spectral function at long wavelengths and low energies in units of m/λ^2 . The solid line corresponds to the noninteracting dispersion. m is the electron effective mass and $\lambda = me^2/\epsilon$ is proportional to q_{TF} the Thomas-Fermi screening wave vector. The quasiparticle width (or scattering rate) is proportional to its energy and the quasiparticle spectral weight vanishes logarithmically at low energies. See text for details.

glecting trigonal warping the band hamiltonian for balanced bilayer graphene is

$$\hat{\mathcal{H}} = \sum_{\vec{k}, \alpha} \frac{\vec{k}^2}{2m} \hat{\psi}_{\vec{k}}^{\alpha, \dagger} [\tau^z \otimes (\vec{\sigma} \cdot \hat{n}_{\vec{k}})] \hat{\psi}_{\vec{k}}^{\alpha}, \quad (1)$$

where the Pauli matrix τ^z acts on the two-degenerate (K and K') valleys, \vec{k} is two-dimensional envelope function momentum measured from the two nodal points K and K' , σ^1 , and σ^2 are Pauli matrices that act on bilayer graphene's pseudospin (layer) degrees of freedom, and $\alpha = \uparrow, (\downarrow)$ accounts for the spin degrees of freedom. The chirality of bilayer graphene's chiral 2DEG is captured by the unit vector $\hat{n}_{\vec{k}} = (\cos 2\varphi_{\vec{k}}, \sin 2\varphi_{\vec{k}})$ where $\varphi_{\vec{k}} = \tan^{-1}(k_y/k_x)$. In Eq. (1) the field operator $\hat{\psi}_{\vec{k}}^{\alpha, \dagger} = (\hat{\phi}_{K+\vec{k}, t}^{\alpha, \dagger}, \hat{\phi}_{K+\vec{k}, b}^{\alpha, \dagger}, \hat{\phi}_{K'+\vec{k}, b}^{\alpha, \dagger}, \hat{\phi}_{K'+\vec{k}, t}^{\alpha, \dagger})$ is a four-component spinor where the low-energy sites³ are the top (t) and bottom (b) layer sites without a near neighbor in the opposite layer. The effective mass is determined by $m = \gamma_1/2v^2 \sim 0.054m_e$ where v is the single-layer Dirac velocity and $\gamma_1 \sim 0.4$ eV (Refs. 3 and 10) is the interlayer hopping amplitude.

The interaction contribution to the bilayer graphene's hamiltonian is layer dependent,

$$\mathcal{V} = \frac{1}{2L^2} \sum_{\vec{q}} \sum_{\alpha, \alpha'} [v_+(q) \hat{\rho}_{-q}^{\alpha} \hat{\rho}_q^{\alpha'} + 2v_-(q) \hat{S}_{-q, \alpha}^z \hat{S}_{q, \alpha'}^z], \quad (2)$$

where $\hat{\rho}_q^{\alpha} = \sum_{\vec{k}} \hat{\psi}_{\vec{k}+\vec{q}}^{\dagger \alpha} \hat{\psi}_{\vec{k}}^{\alpha}$ is the total density per spin, $\hat{S}_{q, \alpha}^z = 1/2(\hat{\rho}_{q, \alpha}^{\uparrow} - \hat{\rho}_{q, \alpha}^{\downarrow})$ is the z component of the corresponding pseudospin density operator in the K valley, with $\hat{S}_{-q}^z \rightarrow -\hat{S}_q^z$ in the K' valley, and v_{\pm} are the symmetric and antisymmetric combinations of the interaction potentials in the same (different) layers $v_s = 2\pi e^2/\epsilon q (v_d = v_s e^{-qd})$ with the layer separation $d = 0.334$ nm.

III. RESPONSE FUNCTIONS

The finite temperature noninteracting response functions can be written as

$$\chi_{\alpha\alpha'}^0(q, i\Omega_n) = -g \sum_{ss'} \int \frac{d^2k}{(2\pi)^2} \frac{n_F[\epsilon_s(\vec{k})] - n_F[\epsilon_{s'}(\vec{k} + \vec{q})]}{i\Omega_n + \epsilon_s(\vec{k}) - \epsilon_{s'}(\vec{k} + \vec{q})} \mathcal{F}_{\alpha\alpha'}^{ss'}(\vec{k}, \vec{k} + \vec{q}), \quad (3)$$

where $g=4$ accounts for the spin and valley degeneracy, $\alpha = \rho, \mathcal{S}^z$ denotes the density and pseudospin density response functions, $s, s' = \pm$ is the chiral band index, $\epsilon_s(\vec{k}) = sk^2/(2m)$, $n_F[\epsilon_s(\vec{k})]$ is the Fermi-Dirac distribution and $\theta_{\vec{k}, \vec{k}+\vec{q}}$ is the angle between the wave vectors \vec{k} and $\vec{k} + \vec{q}$. The angular dependent matrix element for the density-density response $\mathcal{F}_{\rho\rho}^{ss'}(\vec{k}, \vec{k} + \vec{q}) = 1/2(1 + ss' \cos 2\theta_{\vec{k}, \vec{k}+\vec{q}})$ is different from the pseudospin density response $\mathcal{F}_{\mathcal{S}^z\mathcal{S}^z}^{ss'}(\vec{k}, \vec{k} + \vec{q}) = 1/2(1 - ss' \cos 2\theta_{\vec{k}, \vec{k}+\vec{q}})$ due to the presence of the pseudospin operator $\hat{S}_q^z \propto \tau^z$ in Eq. (2). Physically these form factors determine the relative weight of the interband ($ss' = -1$) and in-

traband ($ss' = +1$) excitation contribution to the density-density and pseudospin density response functions.

For undoped bilayer graphene where the Fermi energy lies at the neutrality point (i.e., $\epsilon_F = 0$), the zero-temperature density-density response is completely determined by interband excitations where the product $ss' = -1$. Our low-energy theory has a natural high-energy momentum cutoff given by the bandwidth $k_c = \sqrt{2m\gamma_1}$, from dimensional analysis of Eq. (3) it is clear that the zero-temperature density-density response function $\chi_{\rho\rho}^0 = gm/(2\pi)\Phi_{\rho\rho}(q/k_c, mi\Omega_n/q^2)$, where $gm/(2\pi)$ is the constant density of states for parabolic dispersion, and $\Phi_{\rho\rho}$ is a dimensionless scaling function. The density-density response function is free of any divergences, and sending the bandwidth $k_c \rightarrow \infty$ just gives $\chi_{\rho\rho}^0 = gm/(2\pi)\Phi_{\rho\rho}(mi\Omega_n/q^2)$. Analytically continuing $i\Omega_n \rightarrow \Omega + i\eta$ the respective real and imaginary parts of scaling function $\Phi_{\rho\rho}$ are¹¹ ($y = m\Omega/q^2$),

$$\begin{aligned} \mathcal{Re}\Phi_{\rho\rho}(y) &= \log[4] + \frac{1}{2y} \log \left| \frac{1+2y}{1-2y} \right| - \frac{1}{4y} \log \left| \frac{1+4y}{1-4y} \right| \\ &\quad + \log \left| \frac{1-4y^2}{1-16y^2} \right|, \\ \mathcal{Im}\Phi_{\rho\rho}(y) &= \pi \left(1 - \frac{1}{4y}\right) \Theta\left(y - \frac{1}{4}\right) - \pi \left(1 - \frac{1}{2y}\right) \Theta\left(y - \frac{1}{2}\right), \end{aligned} \quad (4)$$

valid for $\Omega \geq 0$. It is important to note that the above expression is a function of a single scaling variable $y = m\Omega/q^2$, contrary to the case of any two-dimensional system with a Fermi wave vector k_F where it is a function of two variables, namely, $m\Omega/k_F^2$ and q/k_F . This scaling behavior is a consequence of the absence of a Fermi surface for neutral bilayer graphene.

The imaginary part of the density-density response function [$\mathcal{Im}\chi_{\rho\rho}(q, \Omega) = 0$ for $\Omega > q^2/(4m)$], defines the edge of the particle-hole continuum. For interband excitations this is just given by the minimum and maximum values of $\Omega = \epsilon_{\vec{k}+\vec{q}} + \epsilon_{\vec{k}}$. The minimum energy for a particle-hole pair is attained for $\theta_{\vec{k}, \vec{q}} = \pi$, this can be seen by completing the square and writing $\min(\Omega) = 1/m[(k-q/2)^2 + q^2/4]$. The excitation spectrum here $\Omega > q^2/(4m)$ is similar to that of neutral graphene $\Omega > v|q|$ with the difference coming from switch to a parabolic dispersion.

Within the RPA,¹²

$$v_+^{RPA}(q, \Omega) = \frac{v_+(q)}{1 + v_+(q)\chi_{\rho\rho}^0(q, \Omega)}, \quad (5)$$

we recover the static screening [$\Omega \rightarrow 0$ limit of $v^{RPA}(q, \Omega)$] form of Ref. 13, with the Thomas-Fermi wave vector $q_{TF} = g(me^2)/\epsilon \log[4]$. Due to the positive-definite value of $\mathcal{Re}\chi_{\rho\rho}^0(q, \Omega)$, bilayer graphene at the neutrality point excludes any plasmon excitations. The absence of a plasmon mode is not unique for neutral bilayer graphene and is phenomenologically similar to the case of neutral single-layer graphene. In neutral single-layer graphene the density-density response function vanishes inside the particle-hole continuum thereby excluding plasmon excitations, techni-

cally different from the case of bilayer graphene. Using the continuity equation we can relate the optical conductivity to the density-density response function. The RPA optical conductivity (accounting for spin and valley degeneracy),

$$\sigma(\Omega) = 4 \lim_{q \rightarrow 0} \frac{ie^2\Omega}{q^2} \frac{\chi_{\rho\rho}^0(q, \Omega)}{1 + v_q \chi_{\rho\rho}^0(q, \Omega)} = \frac{e^2}{2\hbar}, \quad (6)$$

(restoring \hbar) is purely real and frequency independent similar to the case of single-layer graphene. The optical conductivity is not influenced by interactions within RPA and retains its universal value of $e^2/2\hbar$.

IV. QUASIPARTICLE PROPERTIES AT $T=0$

The essential feature of a Fermi liquid is encoded in the imaginary part of the quasiparticle self-energy ($\mathcal{I}m\Sigma$) or inverse quasiparticle lifetime, which for a homogeneous 2DEG gives $\mathcal{I}m\Sigma \sim (\epsilon - \epsilon_F)^2 \log|\epsilon - \epsilon_F|$. In the case of neutral bilayer graphene the absence of a Fermi surface coupled with the effect of the interband excitations invalidates such a description. In this section we calculate the imaginary part of the quasiparticle self-energy for neutral bilayer graphene and show that $\mathcal{I}m\Sigma \propto \epsilon_{\vec{k}}$ for $\epsilon_{\vec{k}} \rightarrow 0$ indicating non-Fermi-liquid behavior.

First let us examine Fermi's golden rule expression for short-ranged (q independent) interaction u_{eff} . This is a rather crude approximation but as we argue below still gives the essential features of the predicted non-Fermi-liquid behavior. Based on Fermi's golden rule the inverse quasiparticle lifetime can be expressed as

$$\frac{1}{\tau_{+, \vec{k}}} = \frac{gm u_{eff}^2}{2\pi} \int \frac{d^2k'}{(2\pi)^2} \text{Im} \Phi \left(\frac{m(\epsilon_{k'} - \epsilon_k)}{|\vec{k} - \vec{k}'|^2} \right) \cos^2 \theta_{\vec{k}, \vec{k}'}. \quad (7)$$

The ϵ_k dependence can be readily obtained by dimensional analysis which gives $1/\tau_{+, \vec{k}} \propto \epsilon_k$; an exact calculation yields $1/\tau_{+, \vec{k}} = 0.1076 gm u_{eff}^2 \epsilon_k$. The linear energy behavior only depends on the fact that Φ is a function of a single scaling variable $y = m\Omega/q^2$, which is a consequence of the scale invariance of neutral bilayer graphene, and is independent of the detailed behavior of Φ . Bare Coulomb interactions would give a different result, however screening dynamically generates a new scale q_{TF} , below which interactions effectively behave as short ranged. Based on the effects of screening and independence of the inverse lifetime on the function Φ we anticipate non-Fermi-liquid behavior in the long-wavelength limit.

The retarded quasiparticle self-energy within RPA can be written as

$$\Sigma_s^{ret}(k, \omega) = \Sigma_s^{res}(k, \omega) + \Sigma_s^{line}(k, \omega), \quad (8)$$

following the line and residue decomposition of Quinn and Ferrell.¹⁴ The quasiparticle self-energy within RPA remains diagonal in the particle-hole basis. It can be shown that the line contribution is purely real and does not contribute to the imaginary part of $\mathcal{I}m\Sigma_+^{ret}$ which for $(\omega > 0)$,

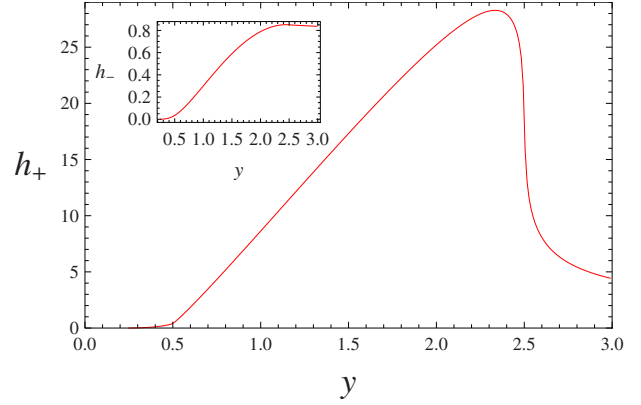


FIG. 2. (Color online) Scaling function $h_+(y)$ for the imaginary part of the electron self-energy with Coulomb interactions in the long-wavelength limit plotted as a function of $y = m\omega/k^2$. The inset shows the scaling function $h_-(y)$ also plotted as a function of $y = m\omega/k^2$. (See text for details.)

$$\begin{aligned} \mathcal{I}m\Sigma_+^{ret}(k, \omega) = & \sum_{s'=\pm} \int \frac{d^2q}{(2\pi)^2} v_+(\vec{k} - \vec{q}) \left(\frac{1 + s' \cos(2\theta_{k,q})}{2} \right) \\ & \times \text{Im} \left[\frac{1}{\epsilon(|\vec{k} - \vec{q}|, \omega - \epsilon(q))} \right] [\Theta(\omega - \epsilon(\vec{q}))], \end{aligned} \quad (9)$$

where $\epsilon(q, \Omega) = 1 + v_+(q)\chi_{\rho\rho}(q, \Omega)$. In the above expression we have neglected the contribution of v_- as it is logarithmically suppressed once screening effects are accounted for within RPA. Dimensional analysis of Eq. (9) implies that $\mathcal{I}m\Sigma_+(k, \omega) = |k|^2 f(m\omega/|k|^2, k/q_{TF})$, where f is a two variable function. Interactions introduce an inverse length scale $\lambda = me^2/\epsilon$ which turns out to be the same order of magnitude as the bandwidth cutoff k_c . In the long-wavelength limit we find that

$$\mathcal{I}m\Sigma_+^{ret}(k \rightarrow 0, \omega \rightarrow 0) = \frac{|k|^2}{g2\pi m} h_+ \left(\frac{m\omega}{k^2} \right) \Theta(4m\omega - k^2). \quad (10)$$

The scaling function $h_+(m\omega/k^2)$ was numerically attained and is plotted in Fig. 2. The theta function in (10) comes from the particle-hole continuum and is independent of the nature of interactions. Using the symmetry relations $\mathcal{I}m\Sigma_s^{ret}(k, -\omega) = \mathcal{I}m\Sigma_{\bar{s}}^{ret}(k, \omega)$ where $\bar{s} = -s$ it is clear to see that $\mathcal{I}m\Sigma_+^{ret}(k, -\omega)$ gives similar expressions as Eq. (10) with a different scaling function $h_-(m\omega/k^2)$ plotted in the inset of Fig. 2. The residue contribution to the real part of the retarded electron self-energy $\mathcal{R}e\Sigma_+^{ret}$ yields a similar expression, however the contribution due to Σ_s^{line} is more singular thereby dominating in the long-wavelength limit. We find that $(\omega > 0)$,

$$\begin{aligned} \mathcal{R}e\Sigma_s^{ret}(k, \omega) = & s \frac{2k^2}{g\pi^2 m} \left(\log \left[\frac{\lambda}{k} \right] \right)^2 \\ & - \frac{4\omega}{g\pi^2} \left(\log \left[\frac{\lambda}{\sqrt{m\omega}} \right] \right)^2 + \dots, \end{aligned} \quad (11)$$

where “...” represent the subleading terms. The expression

for $\omega < 0$ can be attained by exploiting the symmetry relations $\mathcal{R}e\Sigma_s^{ret}(k, -\omega) = -\mathcal{R}e\Sigma_s^{ret}(k, \omega)$. The quasiparticle spectral weight,

$$\lim_{k \rightarrow 0} Z_+ = \frac{1}{1 - \partial_\omega \mathcal{R}e\Sigma_+^{ret}(k, \omega)} \sim \frac{g\pi^2}{4} \left(\log \left[\frac{\sqrt{2m^*} \lambda}{m k} \right] \right)^{-2}, \quad (12)$$

vanishes logarithmically, whereas the effective mass m^* ,

$$\frac{m_+}{m^*} = \frac{1 + m_0 \partial_k^2 \mathcal{R}e\Sigma_+^{ret}(k, \omega)}{1 - \partial_\omega \mathcal{R}e\Sigma_+^{ret}(k, \omega)} \rightarrow 1, \quad (13)$$

remains finite and is not renormalized by interactions. The long-wavelength behavior of the spectral function $A(k, \omega) = A_+(k, \omega) + A_-(k, \omega)$ plotted in Fig. 1 with

$$A_s(k, \omega) = \frac{1}{\pi} \frac{\mathcal{I}m\Sigma_s^{ret}(k, \omega)}{|\omega - \epsilon_k - \mathcal{R}e\Sigma_s^{ret}(k, \omega)|^2 + |\mathcal{I}m\Sigma_s^{ret}(k, \omega)|^2}, \quad (14)$$

was calculated from the leading-order behavior of the electron self-energy and neglecting regular contributions. Symmetry relations for the electron self-energy dictate that $A(k, \omega) = A(k, -\omega)$.

The linear dependence of the $\mathcal{I}m\Sigma$ on the quasiparticle energy predicted above is different from the case of neutral single-layer graphene. In neutral single-layer graphene due to the lack of screening associated with the Dirac point the Fermi velocity develops a logarithmic enhancement.⁵ For neutral graphene this logarithmic velocity enhancement implies that $\mathcal{I}m\Sigma(\omega) \sim \omega/(\log \omega)^2$ which is smaller than $\mathcal{I}m\Sigma(\omega) \sim \omega$. In contrast to single-layer graphene interactions in bilayer graphene are screened, as we have shown and has been pointed out in the literature^{15,16} with Thomas-Fermi screening, that the quasiparticle dispersion in bilayer remains quadratic. Most of our analysis of quasiparticle properties in neutral bilayer graphene has relied on the fact that $\chi_{\rho\rho}^0(q, \Omega) \propto \omega^0$ (i.e., it has a scaling behavior after sending the bandwidth $k_c \rightarrow \infty$). In the next section we identify this regime of non-Fermi-liquid behavior at finite temperature.

V. REGIME OF NON-FERMI-LIQUID BEHAVIOR

Weak interlayer hopping leads to trigonal warping of the band structure in bilayer graphene.³ The temperature associated with this effect can be estimated by calculating the energy scale at which trigonal warping effects compete with the quadratic dispersion kept within our model. Using the bare parameters of graphite $\gamma_3 \sim 0.1\gamma_0$ (Ref. 10) we can estimate that the temperature below which the trigonal warping effect becomes relevant is $T_1 \sim 40$ K. Below this scale the electron dispersion crosses over from quadratic to linear, and the system behaves like single-layer graphene, and our results no longer apply. Recently there has been discussions of interaction-driven spontaneous symmetry breaking in neutral bilayer graphene in the absence of trigonal warping, due to the marginal relevance of weak short-range repulsive interactions.^{7,8} Since the interaction is only marginally relevant, the transition temperature T_c into the possible broken-symmetry phases are exponentially small. The non-Fermi-liquid behavior discussed here thus applies to temperatures above the higher of T_c and T_1 . Our analysis can be extended to finite temperatures, again dimensional analysis dictates that the polarization function have the scaling form $\Phi(m\Omega/k_B T, m\Omega/q^2)$. From a simple scaling analysis of the electron self-energy we conjecture that for temperatures $T > \max(T_1, T_c)$,

$$\mathcal{I}m\Sigma \sim \begin{cases} \omega & \omega > k_B T \\ k_B T & k_B T > \omega \end{cases} \quad (15)$$

indicative of non-Fermi-liquid behavior. In this Rapid Communication we have analyzed the regime $k < q_{TF}$, in the opposite regime $k > q_{TF}$ one can show that the non-Fermi behavior becomes even more pronounced as screening effect is less significant.

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